Ciências ULisboa

5. Numerical Mathematical Morphology

Elementary morphological operations (erosion and dilation). Morphological opening and closing operations. Morphological gradient. Half-gradients. Top-Hat and Bottom-Hat Operations. Numeric geodetic reconstruction. Regional extremes. Watershed Transformation.

Fernando Soares

Introduction

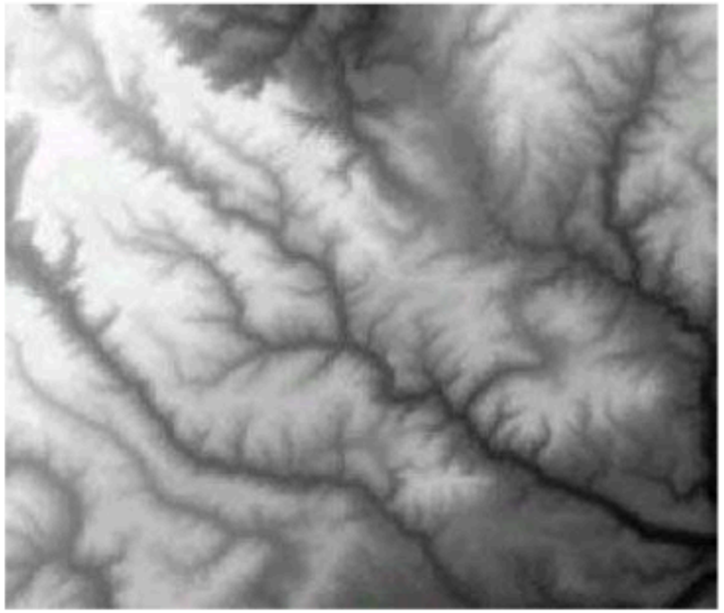
The elementary operations of binary mathematical morphology can be extended to greyscale images through the use of the "**minimum**" and "**maximum**" operations, which parallel the erosion and binary expansion operations.

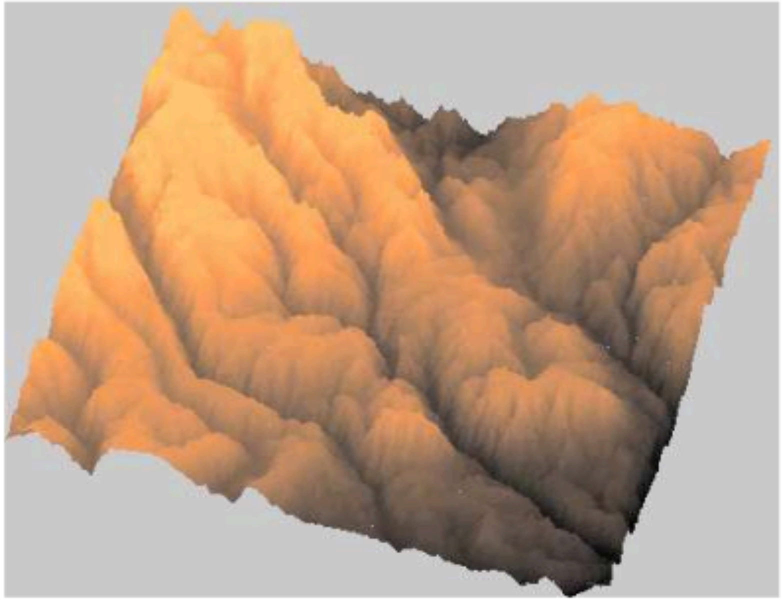
These operations assign each pixel in the image new values corresponding to the minimum or maximum value of a given neighborhood around that pixel. The neighborhood is defined according to the shape of the structuring element.

The numerical mathematical morphology has application in processes of image contrast, texture description, boundary detection and thresholding, among others.

Introduction

In mathematical morphology the pixel intensities of numerical images are considered as topographic elevations.





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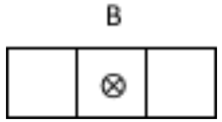
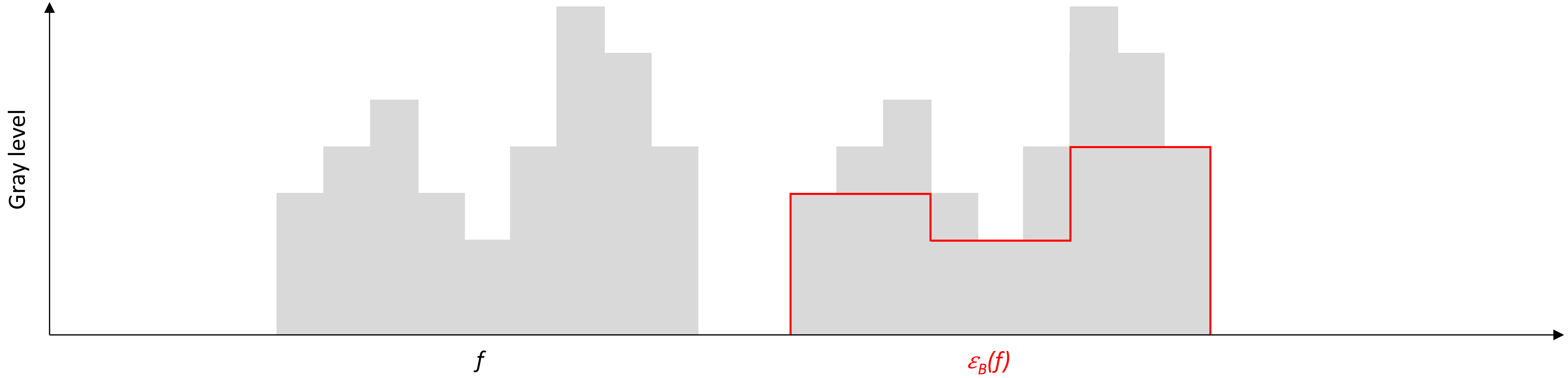
Elementary morphological transformations

The erosion **operation** ε of a given function f by a structuring element B, positioned with its origin at x (B_x), is given by the expression:



$$\varepsilon_B(f(x)) = \min_{b \in B} f(x+b)$$



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Elementary morphological transformations

Example of erosion:

imagem

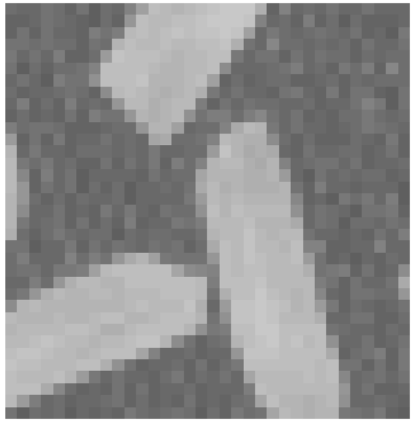
	:	:	:	:	:	
..	15	8	18	6	11	..
..	16	5	21	2	0	..
..	22	14	3	20	19	..
..	4	10	7	1	24	..
..	13	12	17	23	9	..
	:	:	:	:	:	

1	1	1
1	1	1
1	1	1

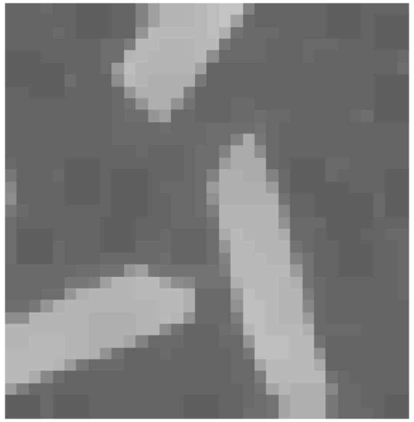
erosão

	:	:	:	:	:	
..						..
..		3	2	0		..
..		3	1	0		..
..		3	1	1		..
..						..
	:	:	:	:	:	

Inicial

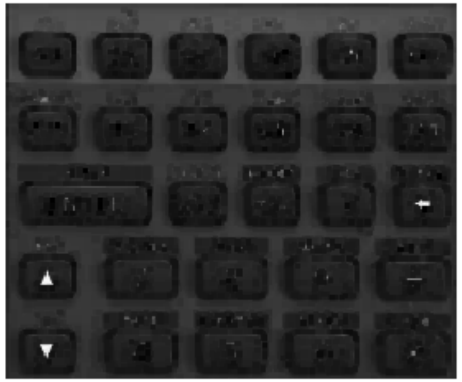


Erosão



Inicial

Erosão



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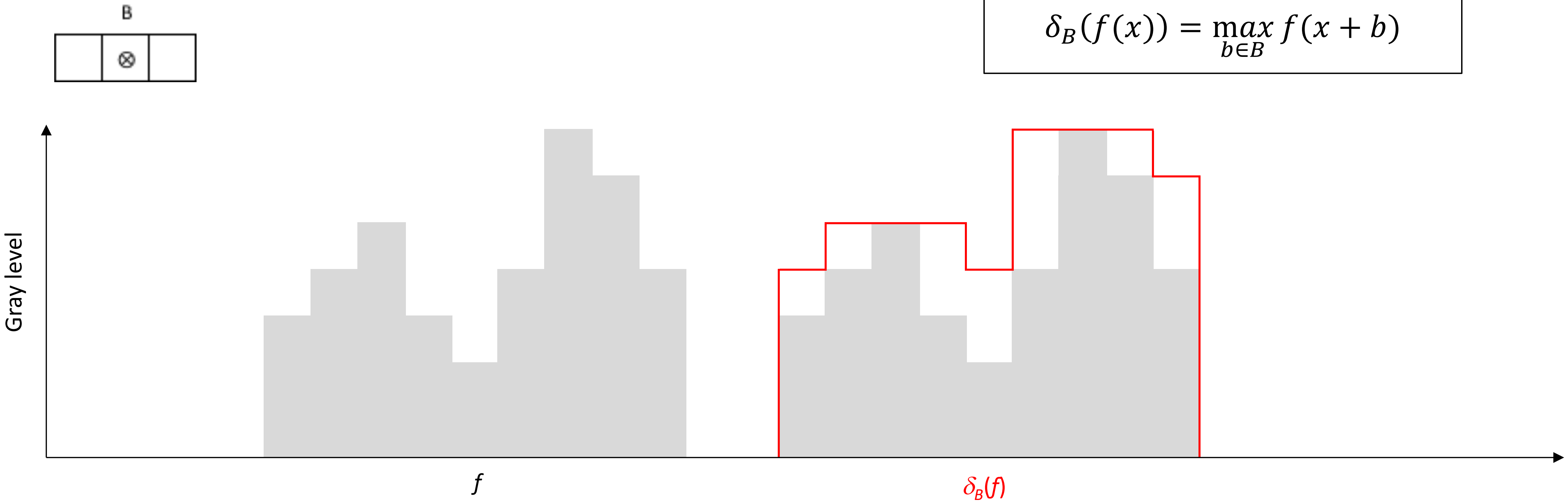
Elementary morphological transformations

The operation of **dilation** δ of a given function f by a structuring element B, positioned with its origin at x (B_x), is given by the expression:

$$\delta_B(f(x)) = \max_{b \in B} f(x+b)$$



Elementary morphological transformations

Example of dilation:

imagem

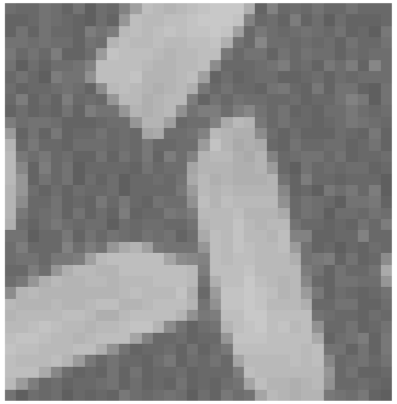
	:	:	:	:	:	
..	15	8	18	6	11	..
..	16	5	21	2	0	..
..	22	14	3	20	19	..
..	4	10	7	1	24	..
..	13	12	17	23	9	..
	:	:	:	:	:	

1	1	1
1	1	1
1	1	1

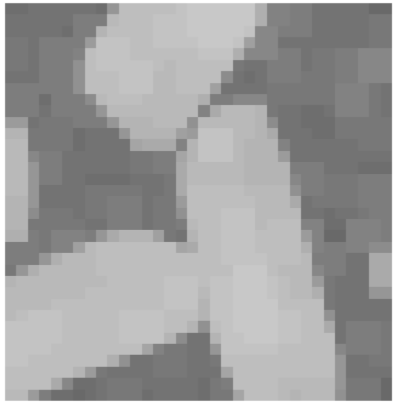
dilatação

	:	:	:	:	:	
..						..
..		24	21	21		..
..		22	21	24		..
..		22	23	24		..
..						..
	:	:	:	:	:	

Inicial



Dilatação



Inicial

Dilatação

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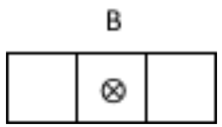
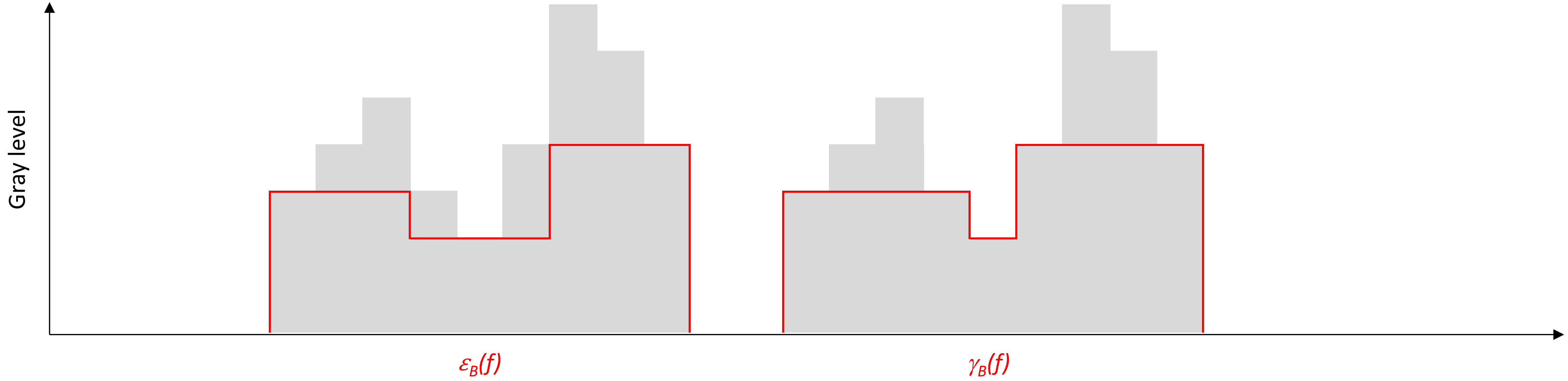
Elementary morphological transformations

The **opening** operation γ of a given function f by a structuring element B, positioned with its origin at *x* (B_x), is given by the expression:



$$\gamma_B(f(x)) = \delta_B\big(\varepsilon_B(f(x))\big)$$



Elementary morphological transformations

Example of opening:

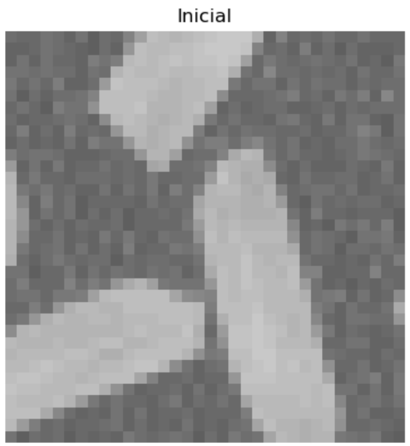
imagem

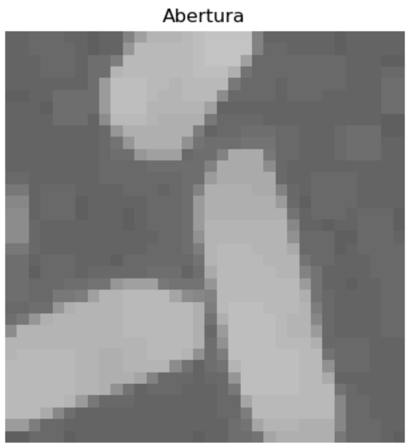
15	8	18	6	11
16	5	21	2	0
22	14	3	20	19
4	10	7	1	24
13	12	17	23	9

1	1	1
1	1	1
1	1	1

abertura

		3		





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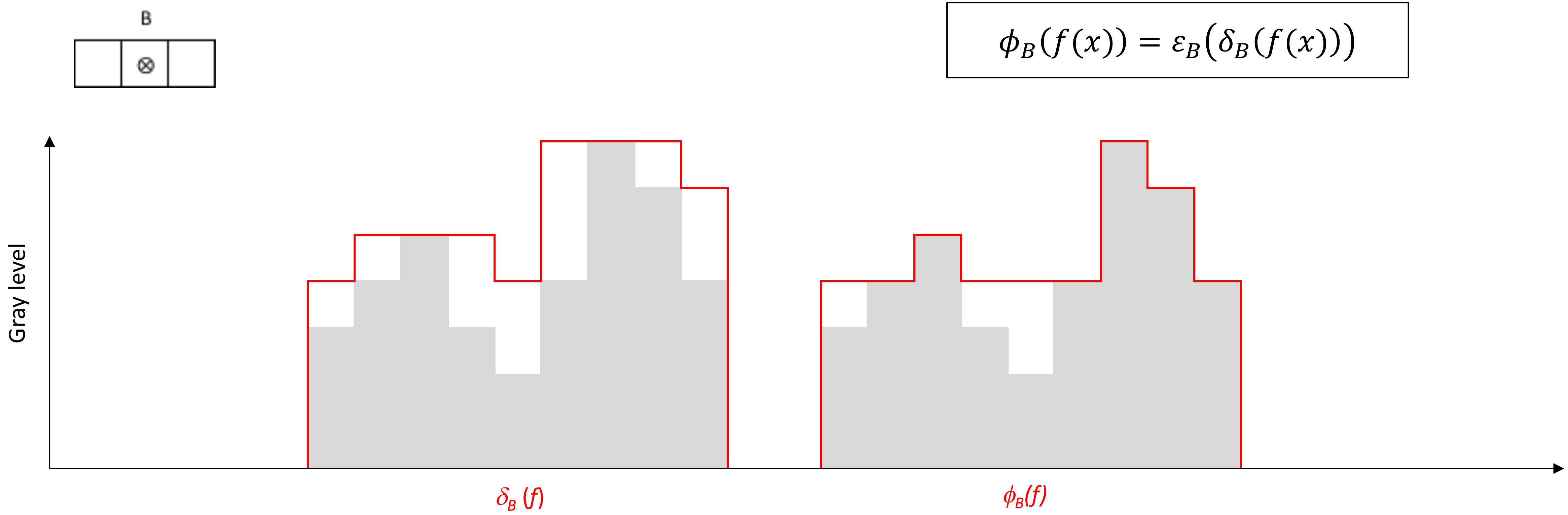
Elementary morphological transformations

The **closing** operation ϕ of a given function *f*, by a structuring element B, positioned with its origin at *x* (B_x), is given by the expression:

$$\phi_B(f(x)) = \varepsilon_B(\delta_B(f(x)))$$



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Elementary morphological transformations

Example of closing:

imagem

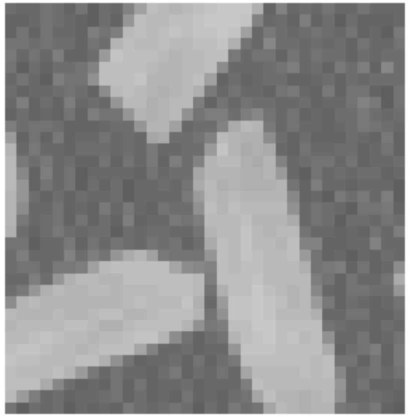
15	8	18	6	11
16	5	21	2	0
22	14	3	20	19
4	10	7	1	24
13	12	17	23	9

1	1	1
1	1	1
1	1	1

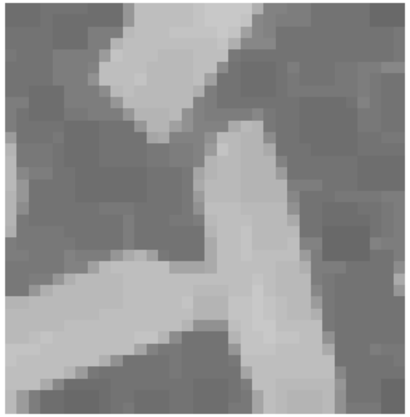
fecho

		21		

Inicial



Fecho



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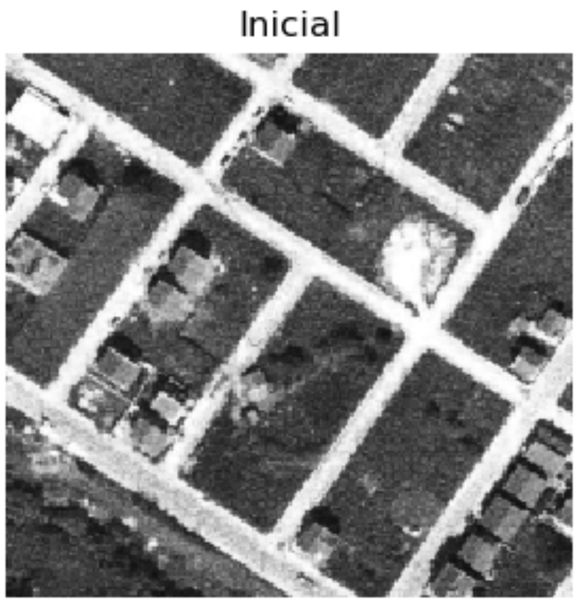
Morphological smoothing

Basic smoothing of an image by morphological methods can be achieved with various approaches. One is to perform an open operation, followed by a close. This removes light and dark artifacts that are at or below the size of the structuring element.

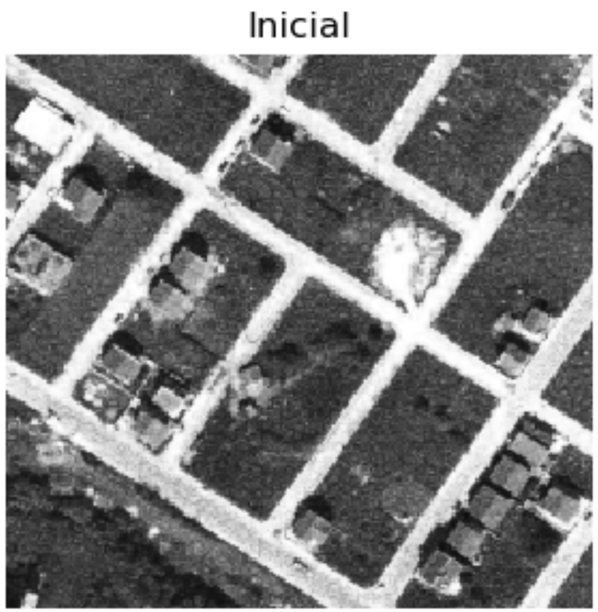


f

$\phi_B(\gamma_B(f))$

Morphological smoothing

A second smoothing approach is to average the erosion and dilation operations of an image.



f

$0.5 \times [\varepsilon_B(f) + \delta_B(f)]$

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Morphological gradient

The common assumption regarding the gradient is that object boundaries, or edges, are located where there are high differences between neighboring pixel values.

Gradient operators are used to highlight these variations.

If there is random noise, the image should be filtered before applying the gradient operator to avoid highlighting the noise as well.

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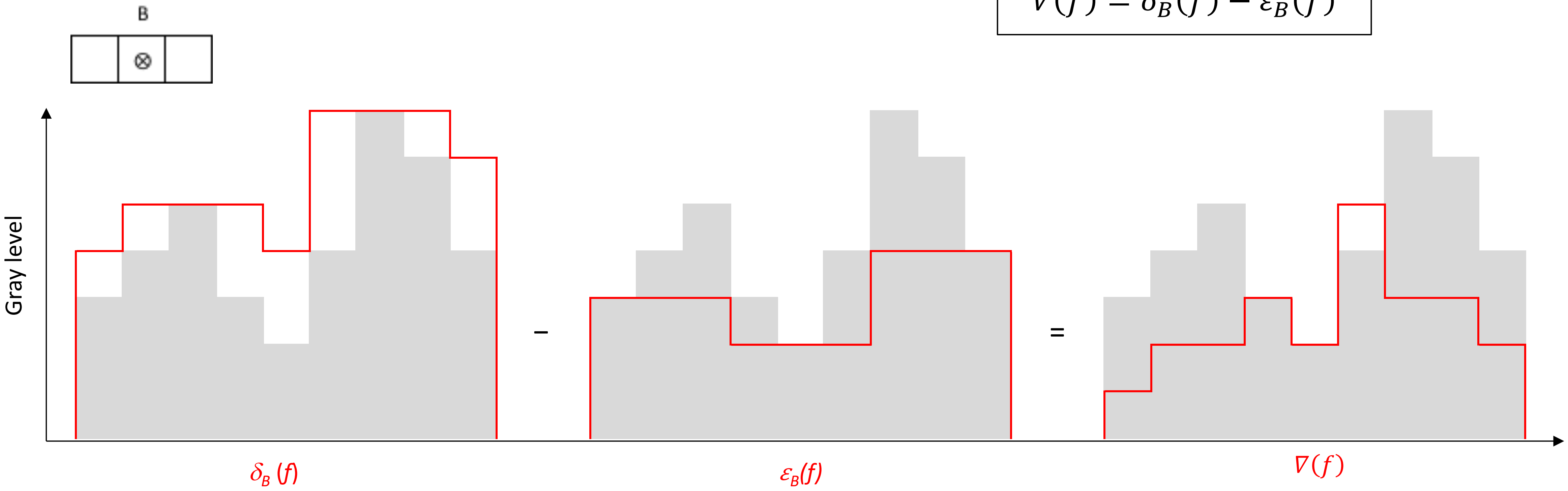
Morphological gradient

The **morphological gradient** ∇ (also called the Beucher gradient) is determined in each pixel by the algebraic difference between dilation $\delta(f)$ and erosion $\varepsilon(f)$.

$$\nabla(f) = \delta_B(f) - \varepsilon_B(f)$$



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Morphological gradient

This operation highlights the most significant transitions in a numerical image.

The result depends directly on the intensity variations in the pixel vicinity defined by a structuring element

Unlike the linear gradient operators of Sobel, Prewitt, or Roberts, the morphological gradients obtained with symmetrical structuring elements tend to depend less on the directionality of object edges.

Morphological gradient

Example of morphological gradient:

imagem

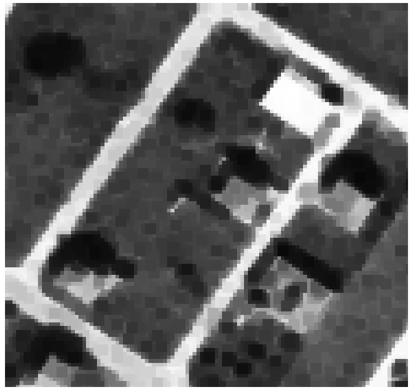
	:	:	:	:	:	
..	15	8	18	6	11	..
..	16	5	21	2	0	..
..	22	14	3	20	19	..
..	4	10	7	1	24	..
..	13	12	17	23	9	..
	:	:	:	:	:	

1	1	1
1	1	1
1	1	1

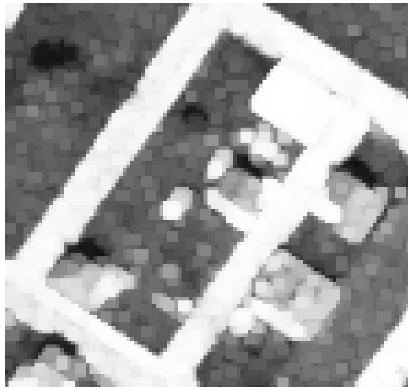
gradiente morfológico

	:	:	:	:	:	
..						..
..		21	19	21		..
..		19	20	24		..
..		21	22	23		..
..						..
	:	:	:	:	:	

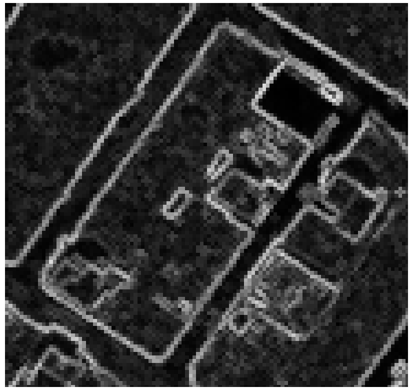
f



$\varepsilon(f)$



$\delta(f)$



$\nabla(f) \equiv \delta(f) - \varepsilon(f)$

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Half gradients

The thickness of an edge detected by a morphological gradient is equal to two pixels: one pixel on each side of the boundary.

Semi-gradients can be used to detect the inner or outer boundaries of a boundary. Such gradients are only one pixel thick.

Half gradients

The **erosion semi-gradient**, or **internal gradient** ∇^-, is defined as the difference between the original image and its numerical erosion.

$$\boxed{\nabla_B^-(f) = f - \varepsilon_B(f)}$$

The internal gradient shows:

- the internal borders of objects that are lighter than the background.
- the external borders of objects that are darker than the background.

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Half gradients

The **dilatation semi-gradient**, or **external gradient** ∇^+, is defined as the difference between the numerical dilatation of an image and its original representation.

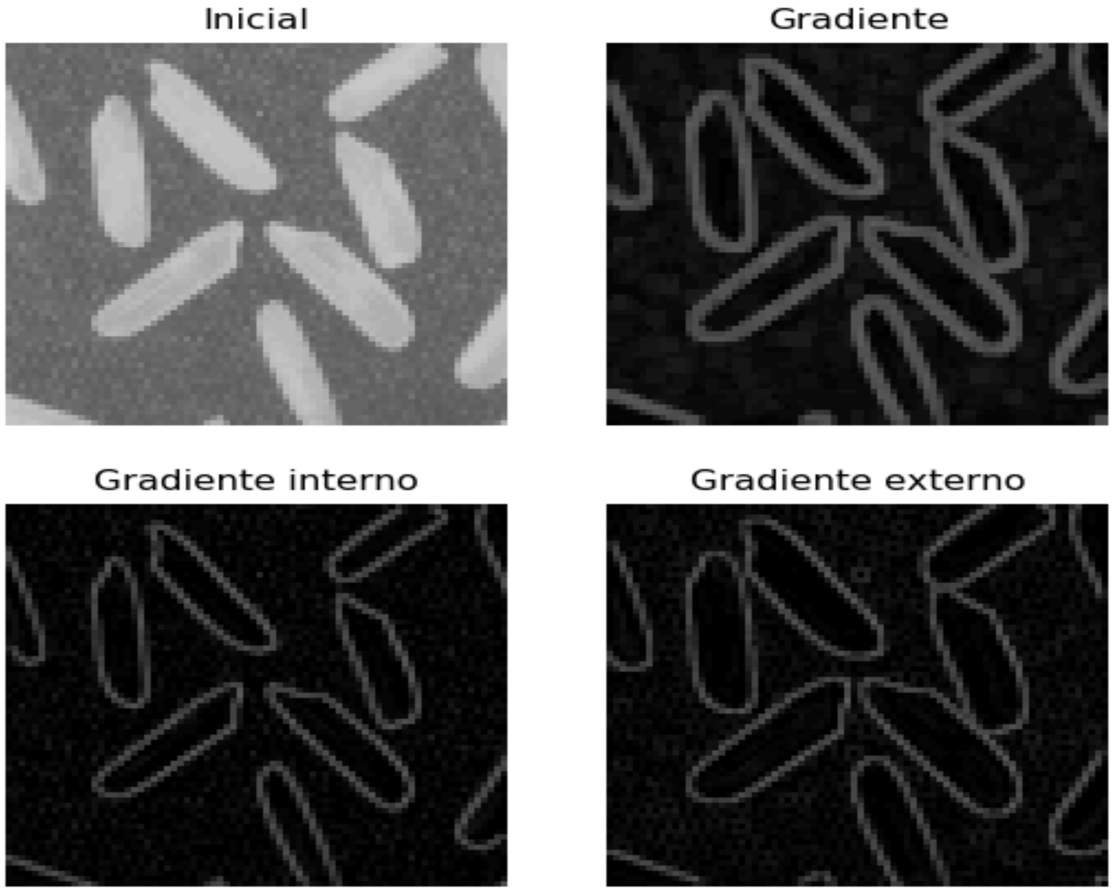
$$\boxed{\nabla_B^+(f) = \delta_B(f) - f}$$

The external gradient shows.

- the internal borders of objects that are darker than the background.
- the external borders of objects that are lighter than the background.

Half gradients

Examples of inner and outer half gradients.



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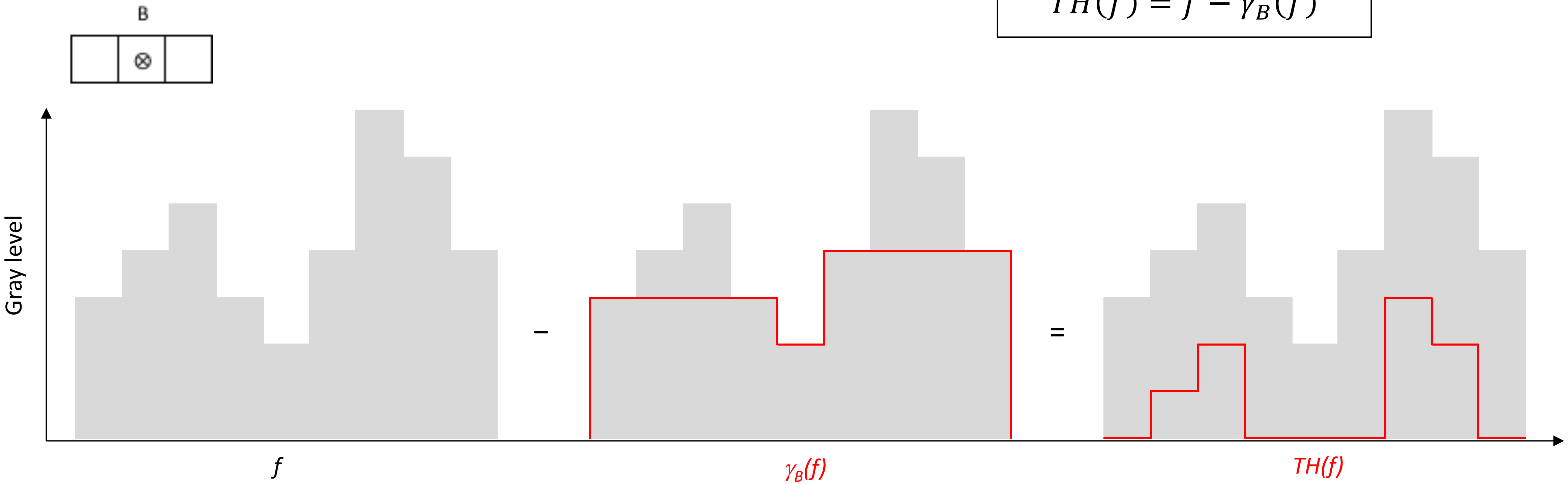
Top-Hat transformation

Top-Hat: consists of the algebraic difference between the function (image) and its opening.

$$TH(f) = f - \gamma_B(f)$$



Top-Hat transformation

The image intensity peaks are extracted.

All structures that do not contain the structuring element are extracted from the image.

Highlights the details of the image.

It is an example of how it is simpler to act on relevant structures rather than directly suppressing the most irrelevant objects.

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Top-Hat transformation

Example of Top-Hat:

Inicial

f

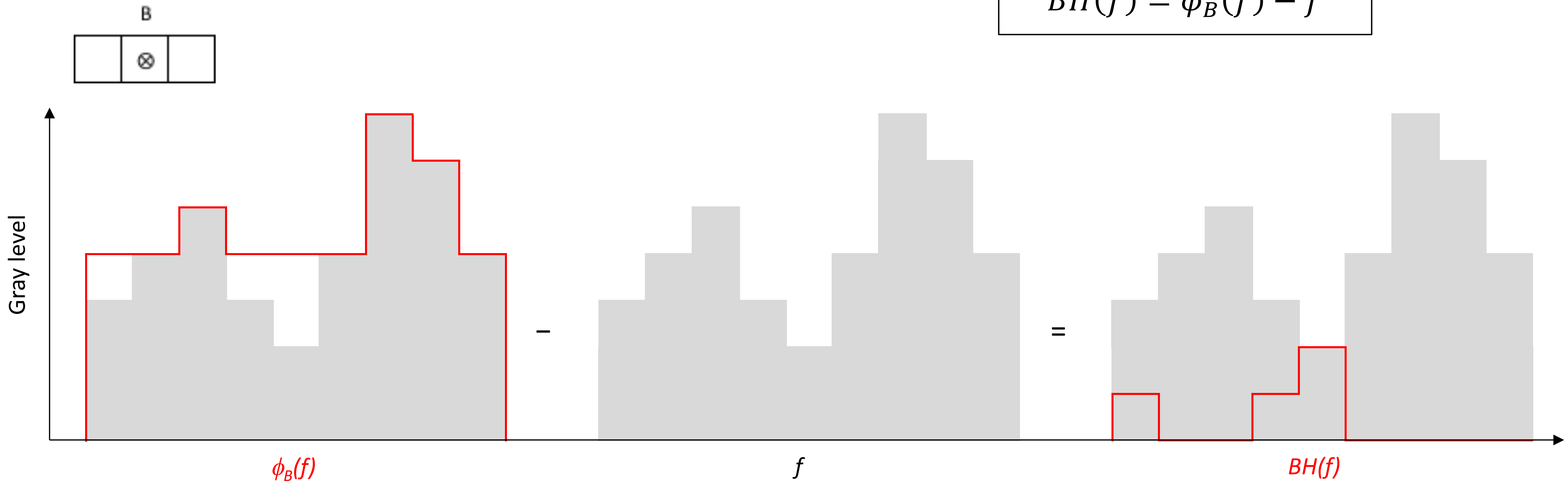
Top-Hat

$TH(f) = f - \gamma_B(f)$

Bottom-Hat transformation

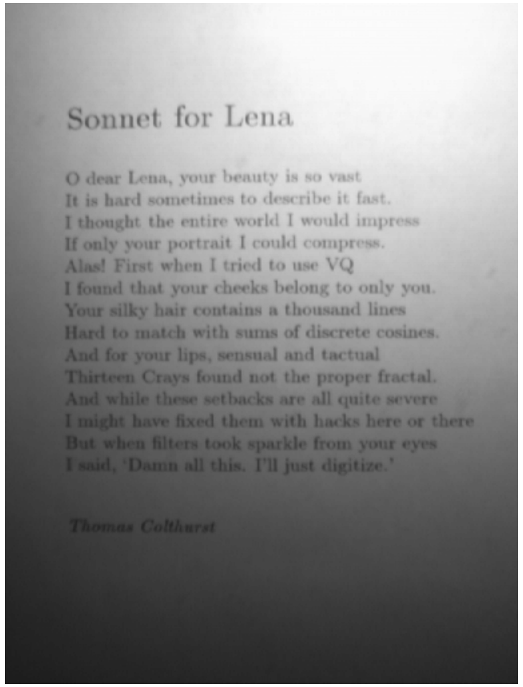
Bottom-Hat: consists of the algebraic difference between the image closing and the initial image.

$$BH(f) = \phi_B(f) - f$$



Bottom-Hat transformation

Example of bottom-hat:

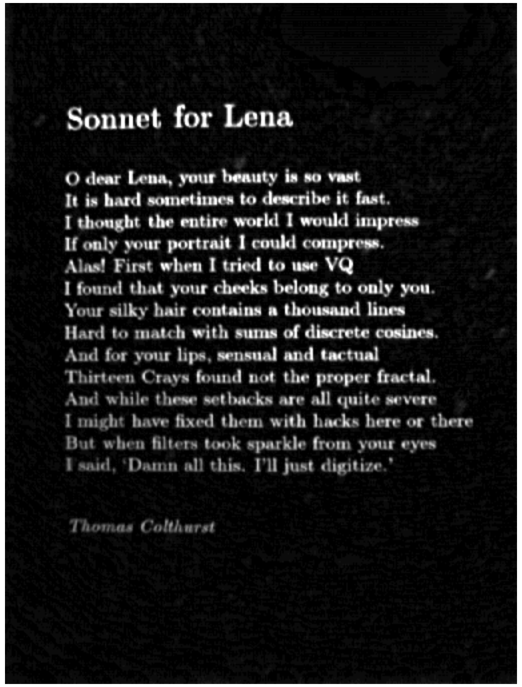


Sonnet for Lena

O dear Lena, your beauty is so vast
It is hard sometimes to describe it fast.
I thought the entire world I would impress
If only your portrait I could compress.
Alas! First when I tried to use VQ
I found that your cheeks belong to only you.
Your silky hair contains a thousand lines
Hard to match with sums of discrete cosines.
And for your lips, sensual and tactual
Thirteen Crays found not the proper fractal.
And while these setbacks are all quite severe
I might have fixed them with hacks here or there
But when filters took sparkle from your eyes
I said, 'Damn all this. I'll just digitize.'

Thomas Colthurst

f



$BH(f) = \phi_B(f) - f$

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Self-complementary Top-Hat

The sum of Top-Hat and Bottom-Hat extracts all objects from the image that do not contain the given structuring element, whatever their relative contrasts (ie peaks and valleys).

From their expressions it can easily be inferred that,

$$\boxed{TH(f) + BH(f) = f - \gamma_B(f) + \phi_B(f) - f = \phi_B(f) - \gamma_B(f)}$$

That is, the self-complementary Top-Hat is given by the difference between closing and opening.

Top-Hat and image enhancement

A simple morphological contrast operator can be obtained with the difference by determining both Top-Hat and Bottom-Hat operators in parallel. Top-Hat is then added to the original image (to highlight lighter objects) and Bottom-Hat is subtracted from the resulting image (to highlight darker objects).

$$\boxed{f + TH(f) - BH(f) = f + f - \gamma_B(f) - \phi_B(f) + f}$$

The resulting values that fall outside the dynamic range of the initial image [z_{min}; z_{max}] will have z_{min}, or z_{max}, depending on whether they are below or above the extremes of the range.

Top-Hat and image enhancement

Example:

Inicial

Top-Hat realce

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Alternating sequential filtering

Filtering of an image that contains dark and bright noise can be achieved by applying a sequence of close-open or open-close operations.

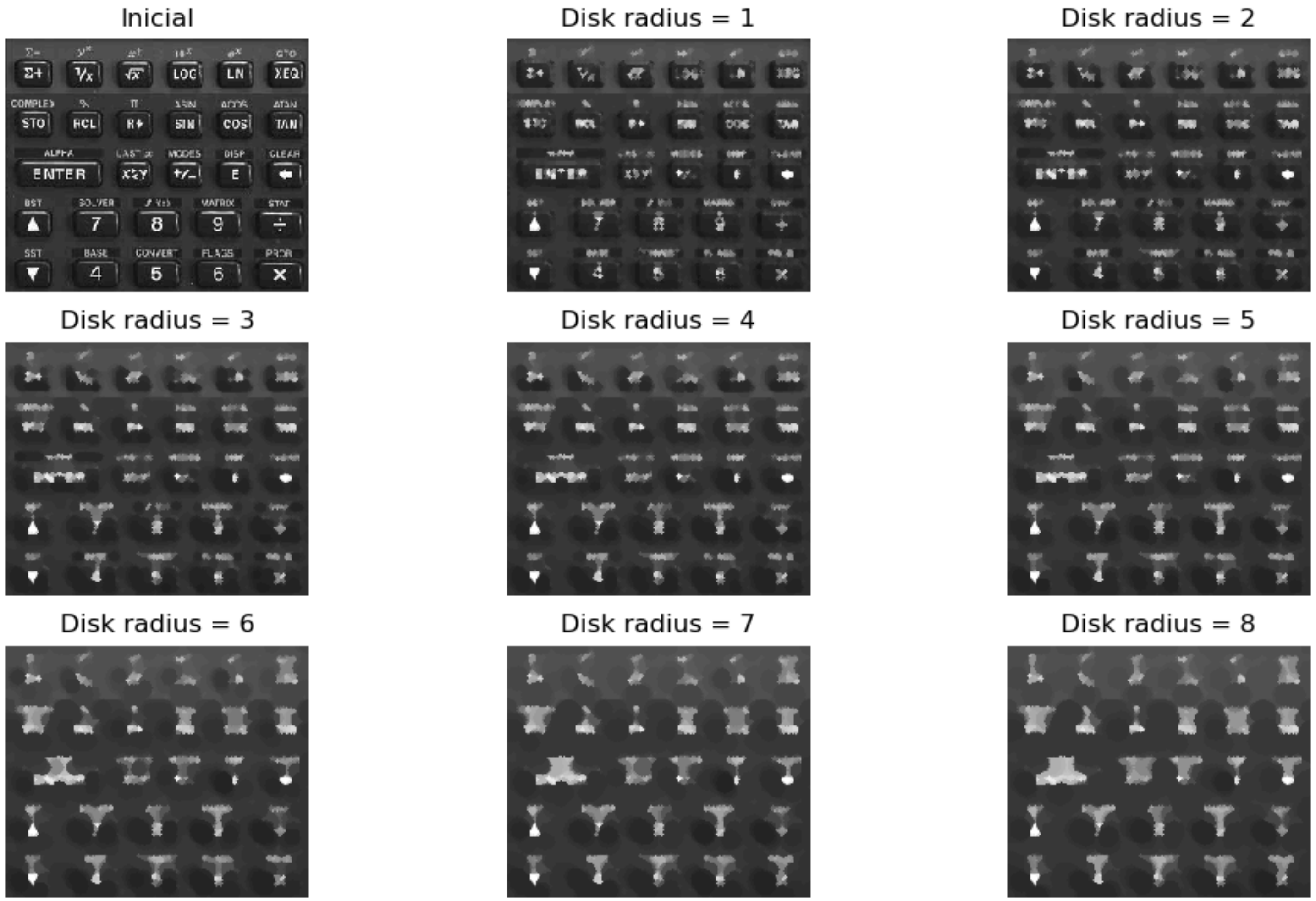
A single sequential close-open, or open-close operation with a large structuring element does not produce acceptable results.

One solution to this problem is to apply reciprocating closures and openings, starting with small structural members and progressively increasing their size to a given final size.

This filtering process by sequential closing-opening, or opening-closing, application is called **alternating sequential filtering**.

Alternating sequential filtering

Example:



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Numeric geodetic transformations

Numerical geodetic transformations are morphological transformations applied to a numerical image *h*, but conditioned by the morphology of another numerical image *f*.

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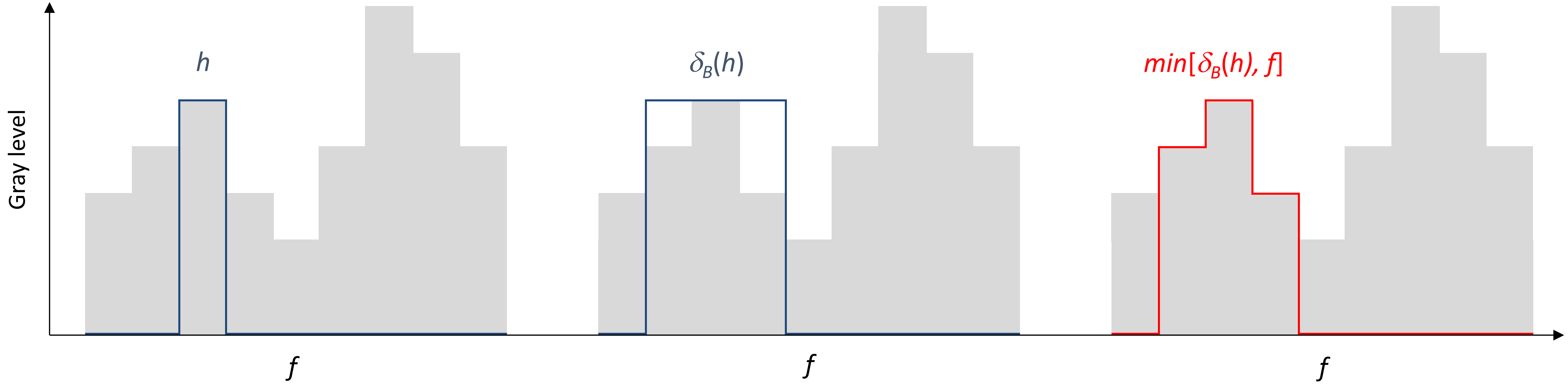
Numeric geodetic transformations

Geodetic dilatation: consists of determining the minimum value between the dilatation of the marker image *h* (≤ *f*) and the function *f*.

$$\boxed{\delta_f(h) = min\left[\delta_B\big(h(x,y)\big); f(x,y)\right]}$$



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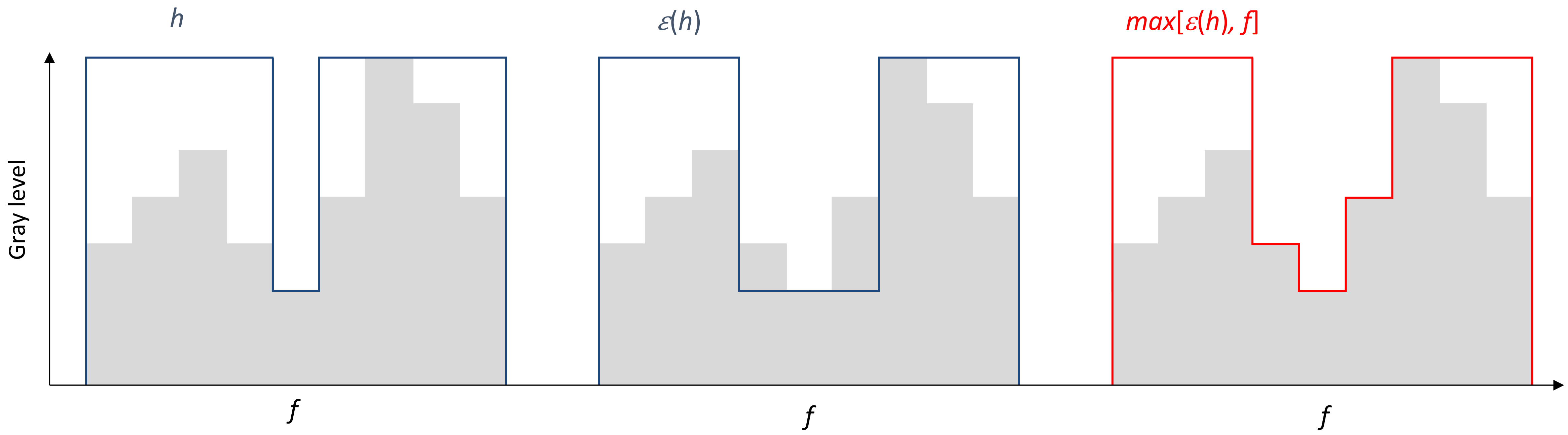
Numeric geodetic transformations

Geodetic erosion: consists in determining the maximum between erosion of the marker image *h* (≥ *f*) and the function *f*.

$$\boxed{\varepsilon_f(h) = max\left[\varepsilon_B\left(h(x,y)\right); f(x,y)\right]}$$



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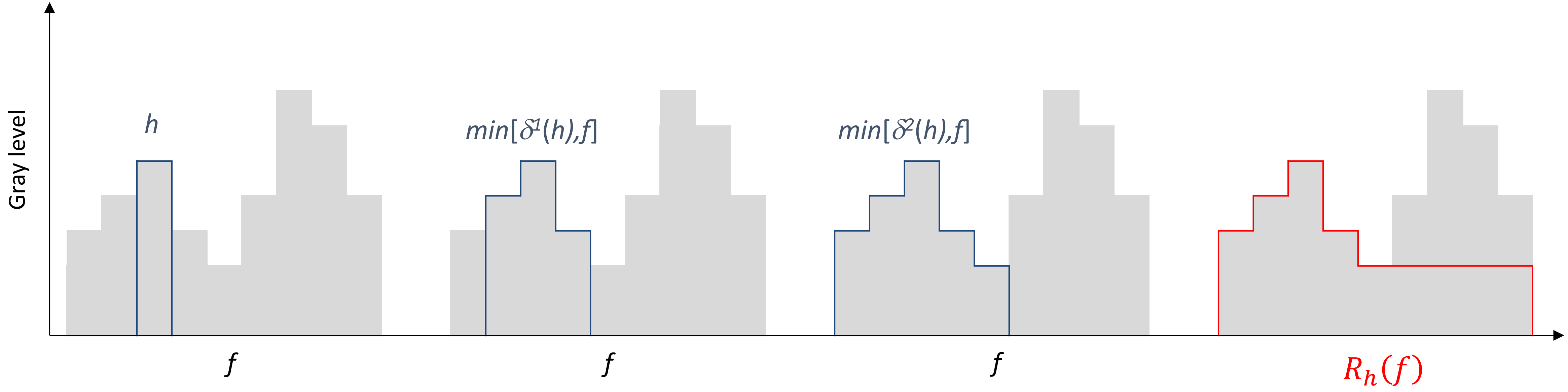
Numeric geodetic transformations

Geodetic reconstruction by successive **geodetic dilations**.

$$\boxed{R_h(f) = \delta_f^{(i)}(h)}$$

until the condition is verified $\delta_f^{(i+1)}(h) = \delta_f^{(i)}(h)$



Numeric geodetic transformations

Example of geodetic reconstruction by successive geodetic dilatations.

Inicial

markers

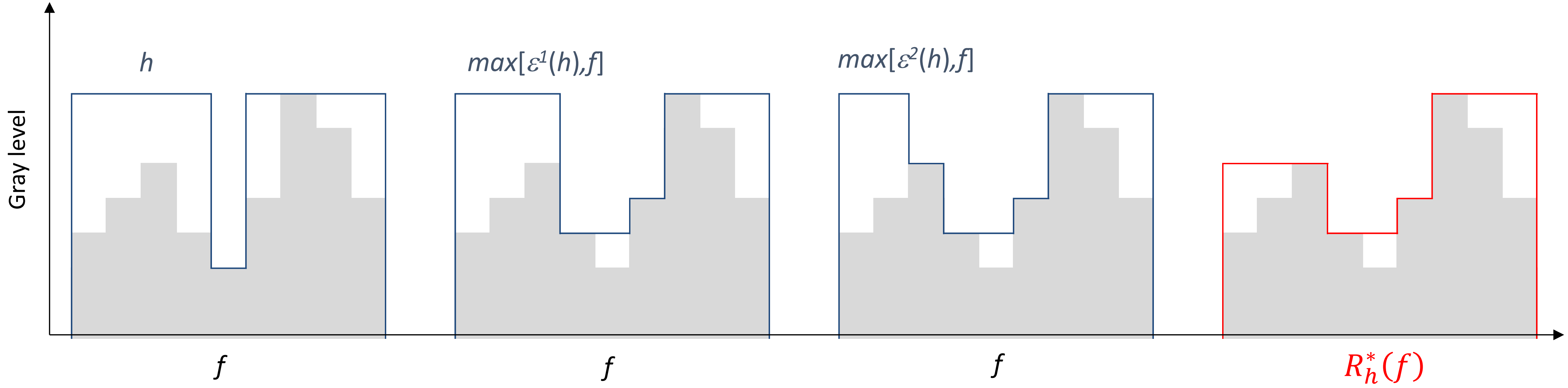
Reconstrução

Numeric geodetic transformations

Numerical **geodetic reconstruction** by successive **geodetic erosions** (Dual Reconstruction).

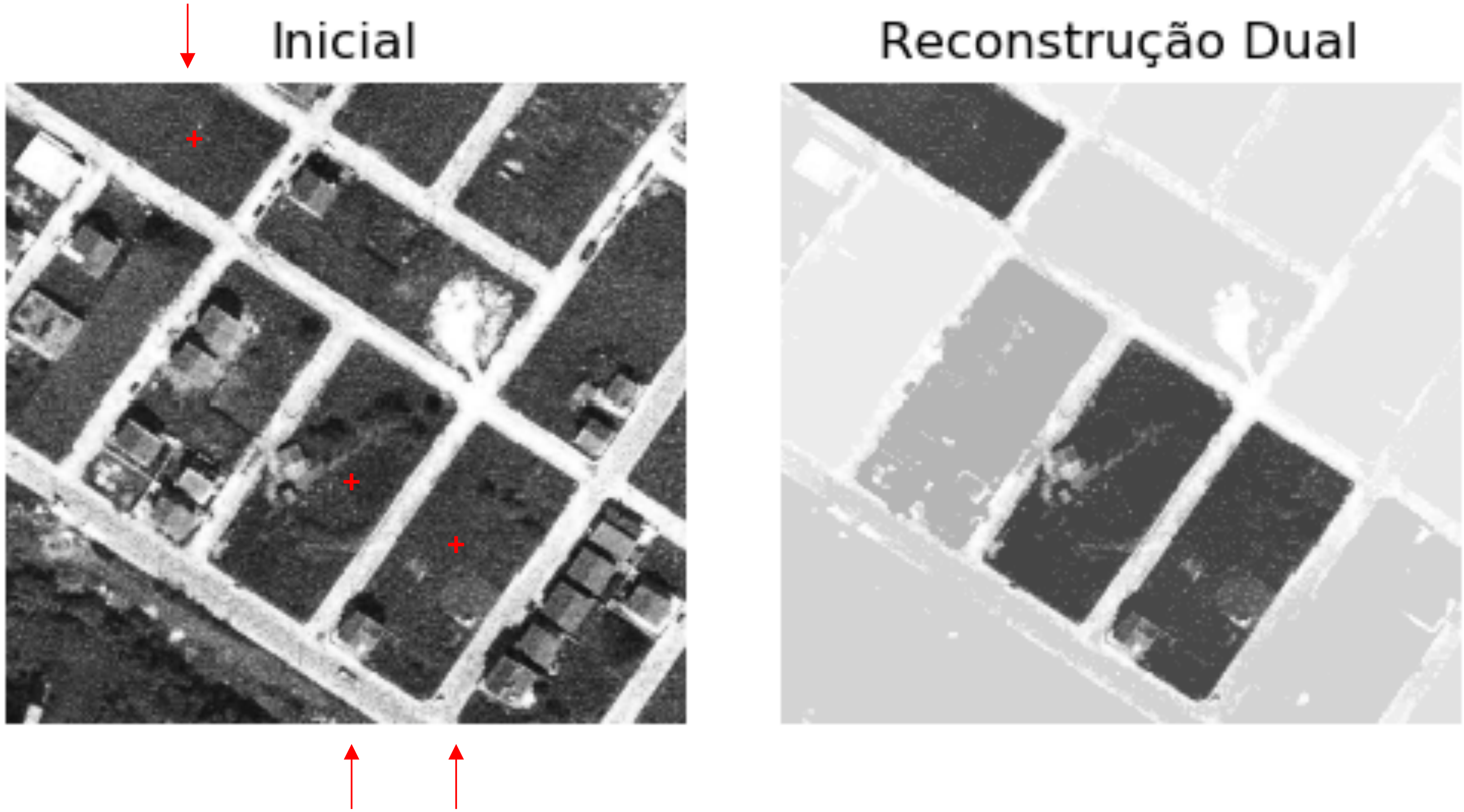
$$\boxed{R_h^*(f) = \varepsilon_f^{(i)}(h)}$$

until the condition is verified $\varepsilon_f^{(i+1)}(h) = \varepsilon_f^{(i)}(h)$



Numeric geodetic transformations

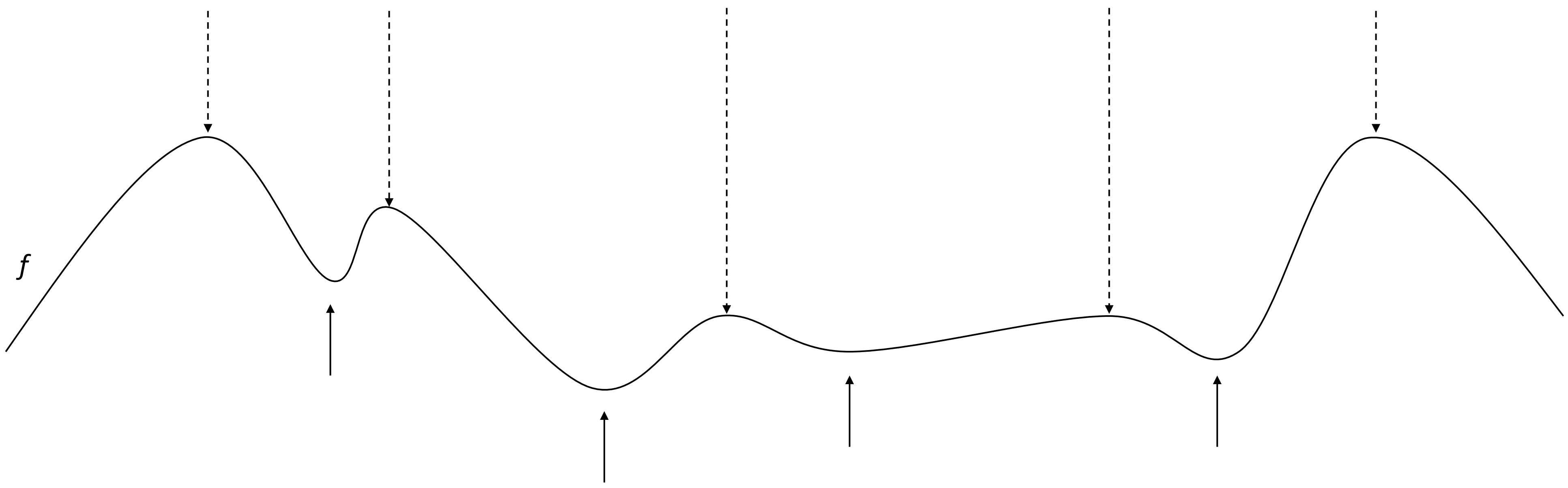
Example of numerical geodetic reconstruction by successive geodetic erosion (Dual Reconstruction).



Regional extremes

The regional minima or maximum images are binary images.

Regional maximum



Regional minimum

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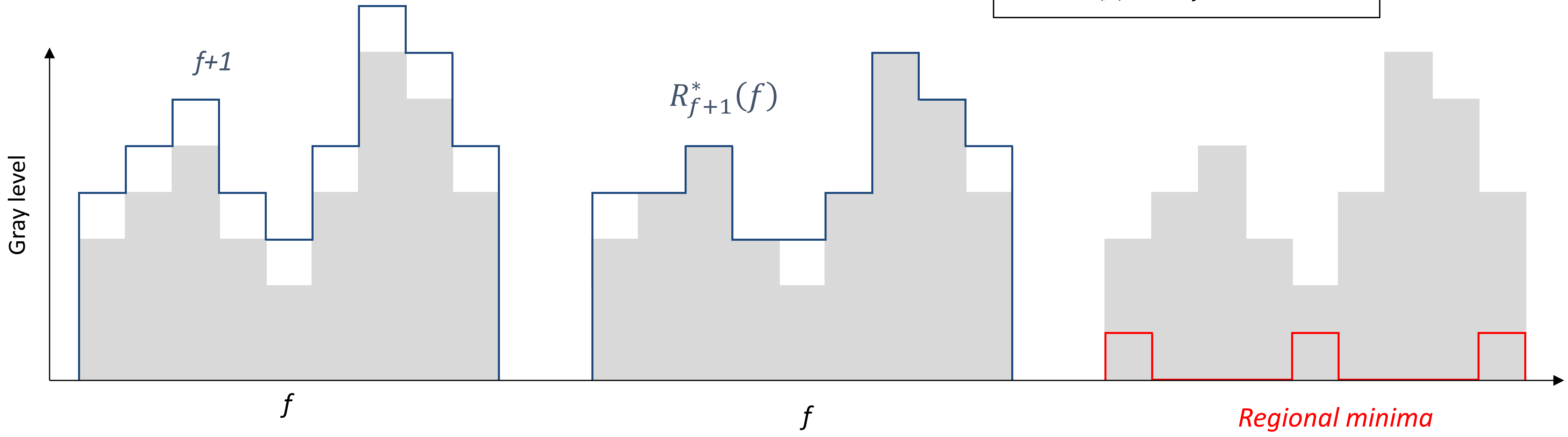
Regional extremes

A binary set M_h of a numerical image *f* is a regional elevation minimum *h* if and only if M_h is a connected surface of equal altitude *h* from which it is impossible to reach a lower elevation point without first having to ascend in function.

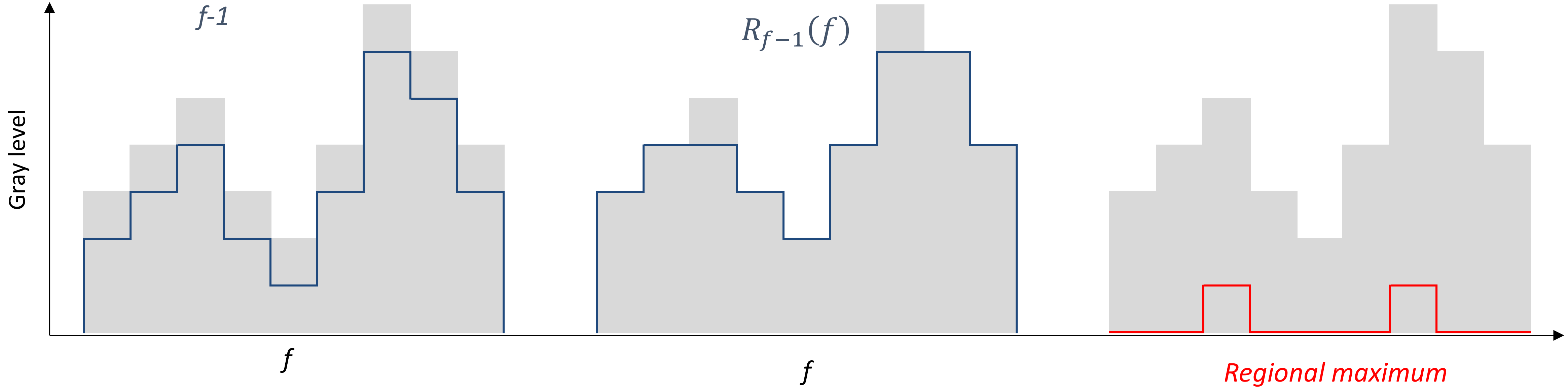
$$\boxed{k_{\min(f)} = R^*_{f+1}(f) - f}$$



Regional extremes

The same applies to the concept of regional maximum *h*, and one has to lower the function before reaching another regional maximum.

$$\boxed{k_{\max(f)} = f - R_{f-1}(f)}$$



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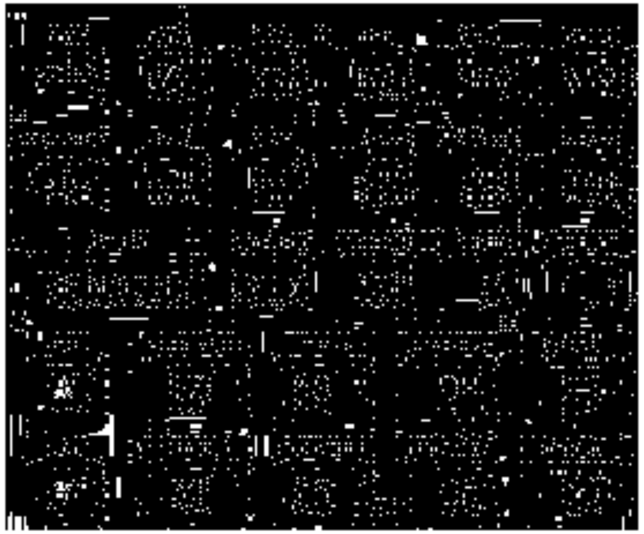
Regional extremes

Example:

Inicial

Minimos

Máximos



Watershed transformation

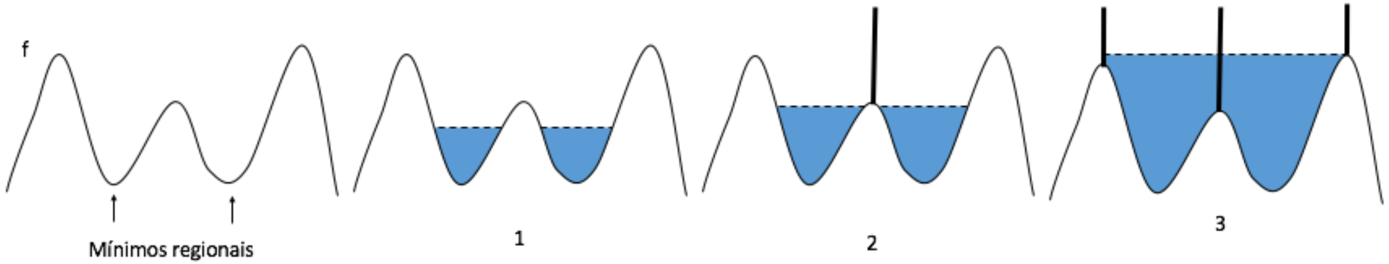
Transformation that aims to **segment a gray function** into distinct regions **from a binary image of markers**:

- Gray Image f: Topographic surface defined by the gray values of the pixels.
- Binary image of markers: These are single pixels or sets of pixels.
- Watershed lines and catchment basins: These are defined by a process analogous to the physical process of vertical surface immersion at constant velocity from a binary image of markers.

Watershed transformation

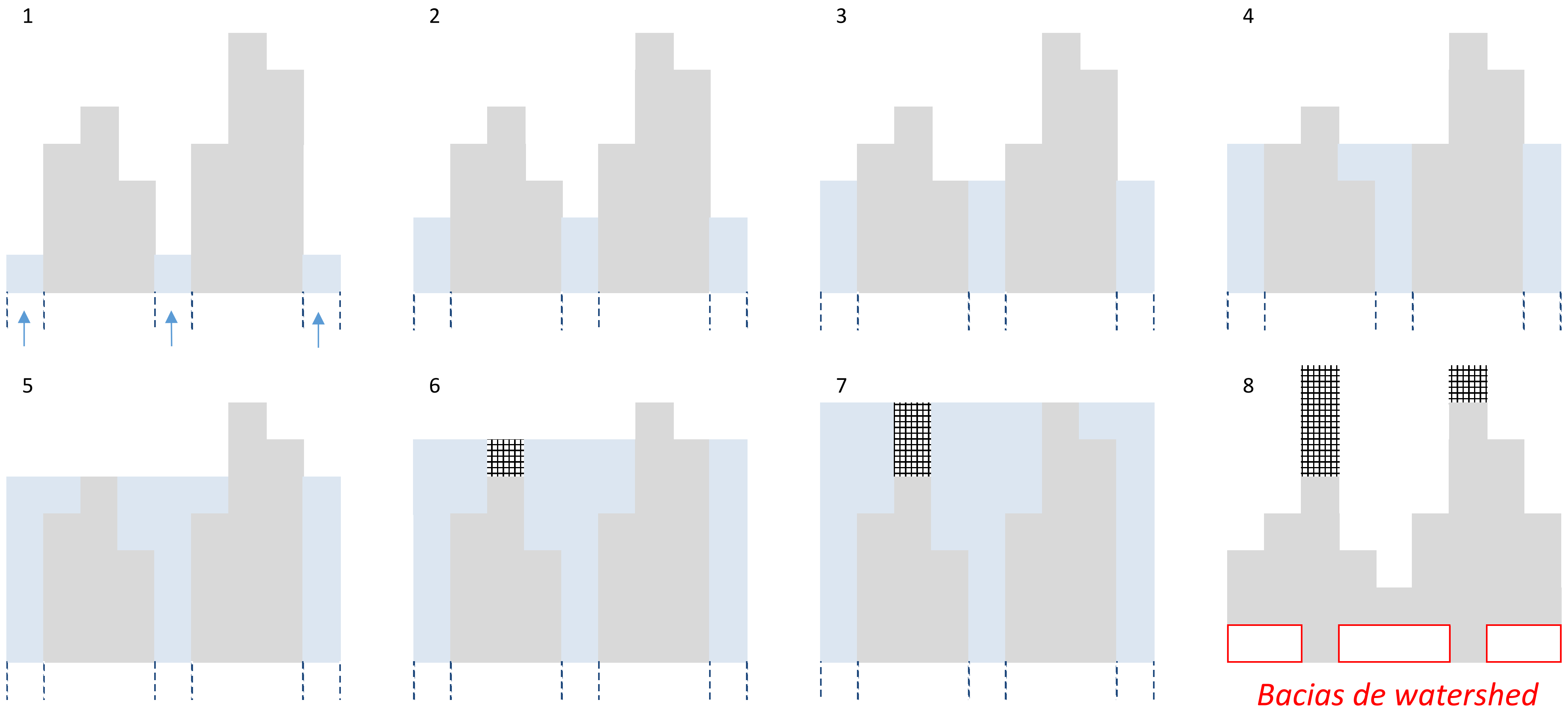
By analogy with a process of immersing a surface with holes in its regional minimums in water, at some point two or more flooded valleys will eventually merge. To prevent this from happening, barriers are raised at all points on the surface where fusion occurs.

At the end of the process, only barriers are represented. These barriers are called surface watershed lines, which separate runoff basins from each other.



Watershed transformation

Illustration of the flooding process from regional minimums.



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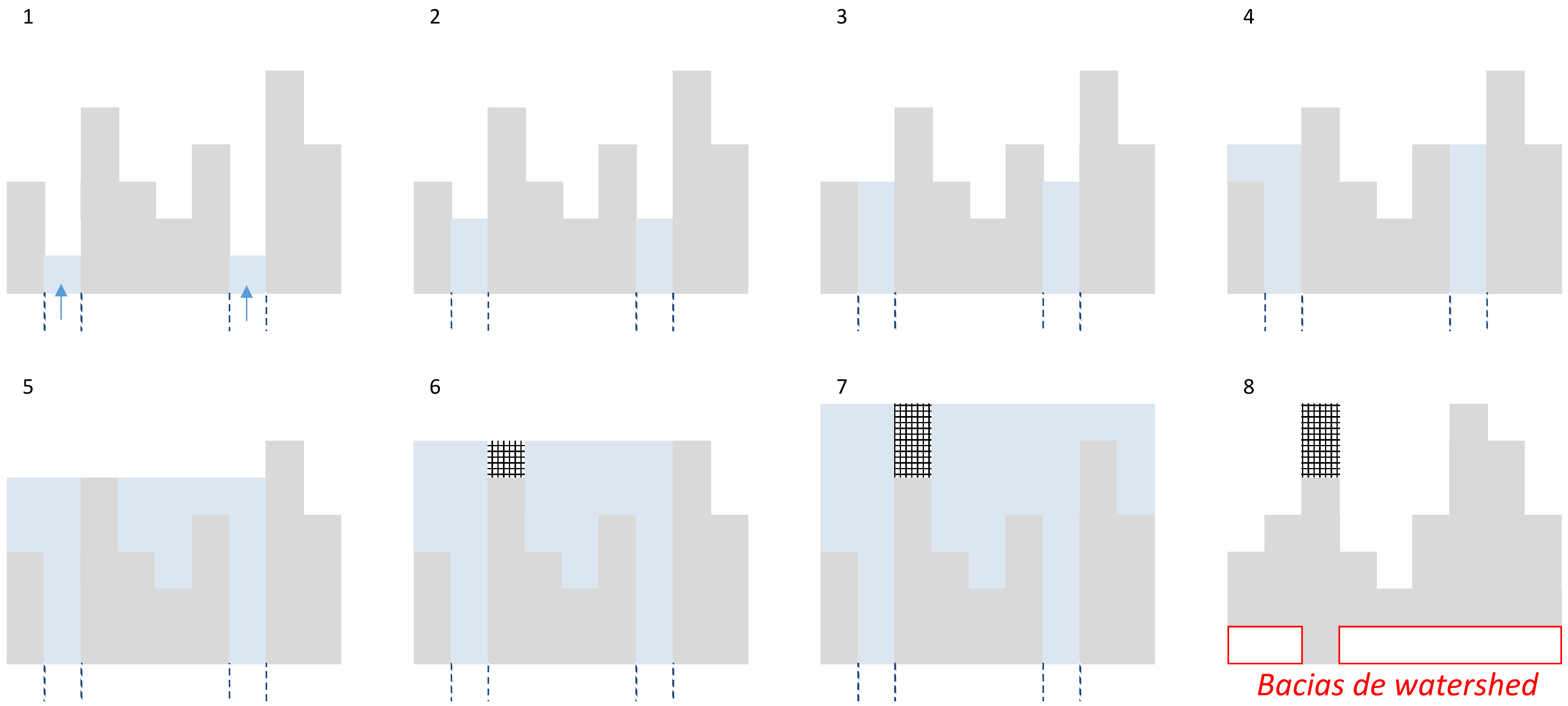
Watershed transformation

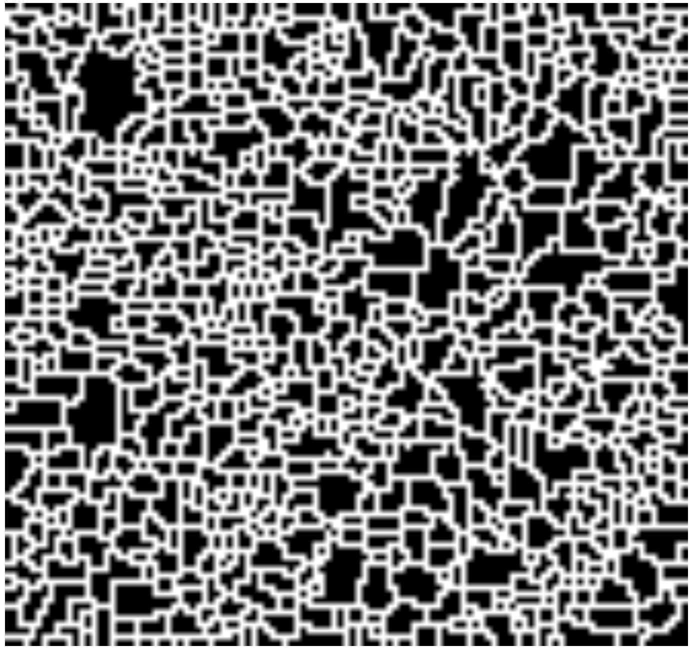
Illustration of the generic flood process from any set of markers.



Watershed transformation

Example of watershed transformation from regional minimums.

Imagem inicial

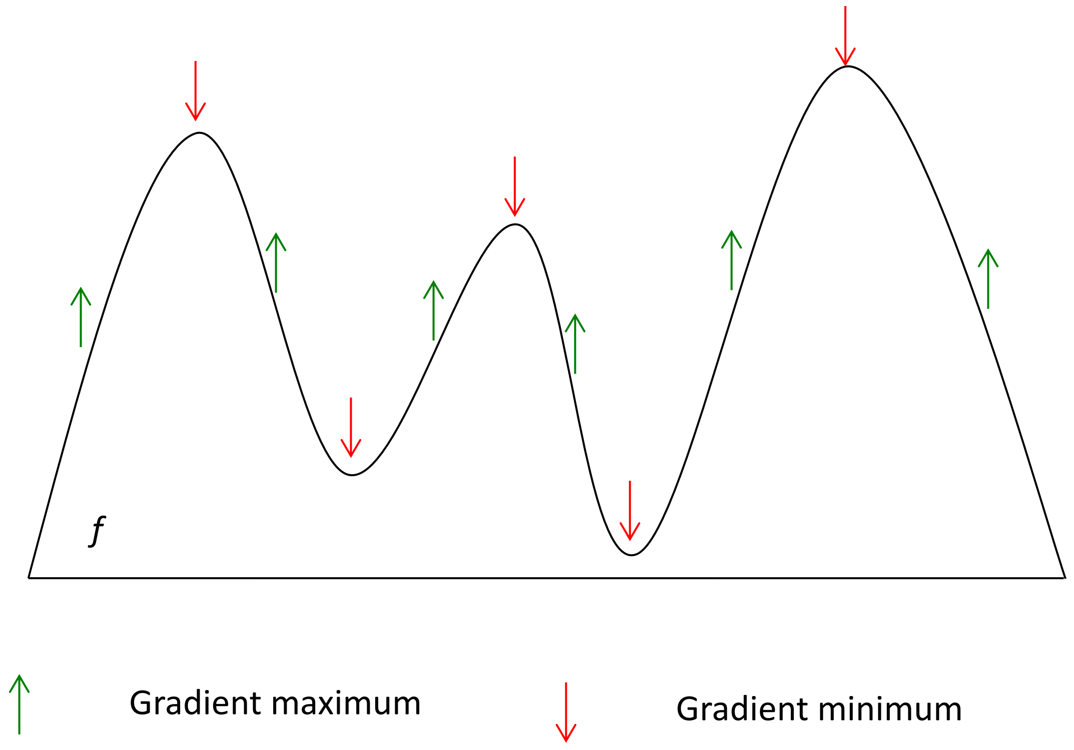


Linhas de *watershed*

Bacias de escoamento

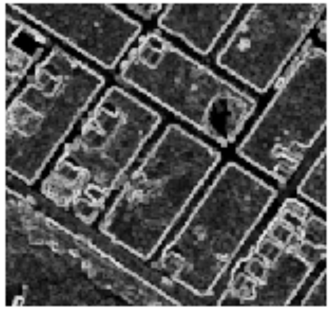
Watershed transformation

Watershed Gradient Image: Instead of the original image f, it applies over the gray image of the morphological ∇ (or other) gradient from the respective regional minima.



Original

Gradiente morfologico



Watershed transformation

Watershed of Gradient Image.